Gasses at High Temperatures

The Maxwell Speed Distribution for Relativistic Speeds

Marcel Haas, 0123323

Summary

In this article we consider the Maxwell Speed Distribution (MSD) for gasses at very high temperatures. So high maybe, that it is not very relevant, but I found it interesting to do. In the MSD the velocities get higher for higher temperatures. At a certain temperature the velocities are distributed around a certain value, with a certain shape given by the MSD. The MSD goes to zero at zero speed and at infinite speed. According to the theory of relativity, particles with a nonzero mass can never reach the speed of light. Therefore, in reality the MSD should go to zero already if the speed reaches the speed of light. First a formula for the relativistic kinetic energy will be given and shown to be useful. Then the MSD for that kinetic energy will be calculated and the results will be plotted for different temperatures to see the effect of the theory of relativity on its shape. In the end the average speed, the root mean square speed and the speed at which the distribution has its maximum will be compared to the 'classical' MSD.

1 Introduction

In gasses, particles move around with all kinds of velocities. The absolute value of the velocity is called speed. These speeds take on a whole range of values, dependent on the temperature. Off course the 'root-mean-square' speed of the particles can be computed by setting the kinetic energy equal to the thermal energy:

$$\frac{1}{2}mv_{rms}^2 = \frac{3}{2}kT \tag{1}$$

$$v_{rms} = \sqrt{\frac{3kT}{m}} \tag{2}$$

Other sorts of averages need to be calculated using the distribution function.

1.1 The Maxwell Speed Distribution

A distribution function, most of times called $D(v)$, is a scalar function. Its actual value has no meaning. What you need it for is to calculate a probability of finding a particle in a certain interval of speeds and it is done by integrating the distribution between those speeds. If we call a probability of finding a particle in the interval between v_1 and v_2 $P(v_1...v_2)$ then you calculate it as follows:

$$P(v_1...v_2) = \int_{v_1}^{v_2} D(v)dv \tag{3}$$

In this way one can see the probability of finding a particle in that particular interval as the area under the function.

It is pretty obvious that the MSD is proportional to the probability of a particle having a velocity $\overrightarrow{v}$ and to the number of vectors $\overrightarrow{v}$ corresponding to a speed v. The probability of a particle having a velocity $\overrightarrow{v}$ is proportional to the Boltzmann factor $e^{-E(s)/kT}$, where for $E(s)$ an appropriate kinetic energy has to be used, and the number of vectors $\overrightarrow{v}$ corresponding to a speed v is proportional to the area of a sphere in the velocity space: $4\pi v^2$, because this space is three dimensional (three perpendicular directions for the vector to lie in), and its radius is v, the absolute value of the vector $\overrightarrow{v}$. Therefore

$$D(v) = C \cdot 4\pi v^2 \cdot e^{-E(s)/kT} \tag{4}$$

In the classical case, $E(s) = \frac{1}{2}mv^2$ and the MSD becomes

$$D(v) = C \cdot 4\pi v^2 \cdot e^{-mv^2/2kT} \tag{5}$$

The factor C has to be determined from normalization (because the particle must have *some* speed):

$$1 = C \cdot 4\pi \int_0^\infty v^2 \cdot e^{-mv^2/2kT} dv \tag{6}$$

From this we find $C = (\frac{m}{2\pi kT})^{\frac{3}{2}}$ and so the MSD without relativistic corrections becomes given by

$$D(v) = (\frac{m}{2\pi kT})^{\frac{3}{2}} \cdot 4\pi v^2 \cdot e^{-mv^2/2kT} \qquad (7)$$

A plot of the MSD for a hydrogen gas at $T = 10.000$ K is shown in Fig 1 where you can clearly see that $D(v) \to 0$ for $v \to 0$ and, less clearly, $D(v) \to 0$ for $v \to \infty$. Of course, these properties are clear from Eq. 7 too.

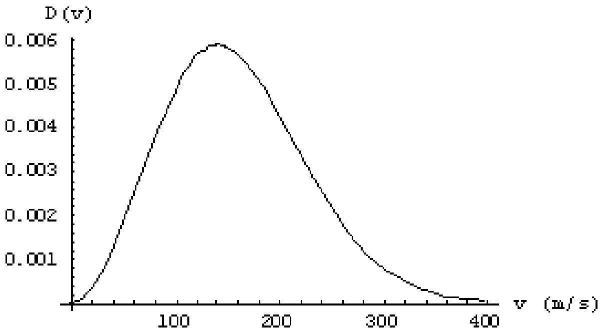

Figure 1: MSD without relativistic corrections for a hydrogen gas at 10.000 K.

The dependance on temperature of the MSD is shown in Figure 2. The highest peak corresponds to the lowest temperature. The peak velocity is temperature dependent as

$$v_{max} = \sqrt{\frac{2kT}{m}} \qquad (8)$$

as can be easily seen by putting the derivative of Eq. 7 equal to 0. The average speed is found by $\overline{v} = \sum vD(v)dv$. This sum can be turned into an integral and it becomes:

$$\overline{v} = (\frac{m}{2\pi kT})^{\frac{3}{2}} 4\pi \int_0^\infty e^{-mv^2/2kT} v^3 dv = \sqrt{\frac{8kT}{\pi m}} \qquad (9)$$

We now have become three kinds of a mean value for the speed, which are all different. In increasing order: the speed at which the MSD has its maximum value, v_{max}, given by Eq 8, the average speed $\overline{v}$ given by Eq 9 and the root-mean-square speed v_{rms}, given by Eq 2.

1.2 The Theory of Special Relativity

According to Maxwell particles in a gas have a whole range of speeds, in theory even extending to very high values. If you increase the temperature high enough, according to the MSD a certain part of the particles will move at speed greater than the speed of light:

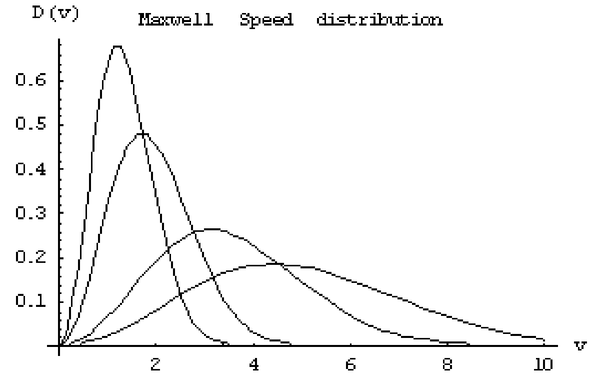

Figure 2: MSD's for different temperatures in random units (they are normalized).

$$N(v \geq c) = N \int_c^\infty D(v)dv \geq 0 \qquad (10)$$

where $N(v \geq c)$ denotes the number of particles with a speed greater than or equal to the speed of light, and N is the total number of particles in our gas. Because there is a lower limit in the integral now, it cannot be calculated analytically anymore.

You can imagine that when the temperature increases to very high values it is possible to reach states where v_{max} becomes in the order of the speed of light $c = 299.792.458$ m/s. The temperatures that we are talking about then are calculated by rewriting Eq 8 like $T = \frac{v_{max}^2 m}{2k}$. If we are talking about a hydrogen gas (hydrogen is the most abundant element in the universe, and I think that if this temperatures are accomplished somewhere, it must be an exotic place in the universe), and we say that v_{max} must be approximately 1/3 of c (say, for simplicity 10^8 m/s) then the temperature must be of order $T = \frac{(10^8)^2 1.68 \cdot 10^{-27}}{2 \cdot 1.38 \cdot 10^{-23}} \approx 6 \cdot 10^{11}$ K. Adapting the MSD to a relativistic description doesn't seem to be that important. The only place I heard about such a temperature is in the iron core of a very massive and highly evolved star, where the temperature exceeds 10^{10} K. Nevertheless it might be very interesting.

1.2.1 An Expression for the Kinetic Energy

In the theory of special relativity one often uses the expression $E_k = \sqrt{p^2c^2 - m^2c^4}$ in which m is the rest mass of a particle. This is indeed a good expression but it is a function of the particles momentum p. This relation is therefore not to be used in the MSD since p doesn't vary linearly with v (the mass of a particle increases with higher speed). Another relation is more useful here. The total energy of a particle (including its rest mass energy) is given by

$$E_{total} = \gamma m c^2 \tag{11}$$

in which

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{12}$$

m is the rest mass of the particle en v and c are the speed of the particle and the speed of light respectively.

To derive an MSD for relativistic speeds, one needs the kinetic energy, while Eq 11 gives the kinetic energy and the rest mass energy together (for $v = 0$, $E_{total} = mc^2$), so in order to obtain a useful expression for the kinetic energy we need to subtract this rest mass energy:

$$E_{kin} = \gamma mc^2 - mc^2 = mc^2(\gamma - 1), \tag{13}$$

which goes to zero for zero speed. This expression for the relativistic kinetic energy will be used to derive the MSD for ultra-high temperatures.

2 The Relativistic Form of the Maxwell Speed Distribution

2.1 The Distribution Function

When we fill in Eq 13 as a function for the kinetic energy in the general form of the distribution, Eq 4, we become an expression for the MSD in which there is taken account of relativistic effects:

$$D(v) = C \cdot 4\pi v^2 \cdot \mathrm{Exp}\Big[\frac{m_0 c^2}{kT}\,\Big(1 - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\Big)\Big] \tag{14}$$

The constant C again has to be determined from normalization,

$$C = 1/\int_0^c 4\pi v^2 \cdot \mathrm{Exp}\Big[\frac{m_0 c^2}{kT}\,\Big(1 - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\Big)\Big] dv \tag{15}$$

The upper integration limit was ∞, but for values higher than c, $D(v)$ must be zero (actually it is imaginary). This integral can unfortunately not be evaluated analytically, so numerical methods will be used.

First we will check whether this distribution function is to be used. Does its value indeed go to zero for $v \to c$? Is its shape for low temperatures like the shape of Fig 1? What happens at higher temperatures? Is the peak ever moving toward higher speeds for higher temperatures as would be expected? Do the peaks smear out too like in the classical form of the MSD?

2.2 Properties of the relativistic MSD

First we will check the limits of the distribution mathematically:

$$\lim_{v \to 0} D(v) = 0 \tag{16}$$

as is easily seen by just filling in this value for v in the distribution. What about the upper limit? All we meant by deriving this equation is to correct for speeds around c, because higher speeds are not possible, according to Einstein. Therefore, it is necessary for $D(v)$ to go to 0 for $v \to c$:

$$\lim_{v \to c} D(v) \propto e^{-\infty} = 0 \tag{17}$$

because the Lorentzfactor ($\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$) goes to infinity when the speed reaches the speed of light. So both of the limits are OK.

Of course, for low temperatures, and hence low speeds, the relativistic MSD should be equal to the 'classical' MSD (at least in first order). To verify this we will just show that the relativistic form of the kinetic energy (Eq 13) for low speeds will become the same as the classical kinetic energy($\frac{1}{2}mv^2$). For small v we can write

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} \tag{18}$$

so

$$mc^2(\gamma - 1) = mc^2(1 + \frac{1}{2}\frac{v^2}{c^2} - 1) = \frac{1}{2}mv^2 \tag{19}$$

for small v. At low temperatures where, according to both the normal and the relativistic MSD, the speeds are low, the relativistic MSD is similar to the one Maxwell derived in 1859. Because if we fill in another expression for the kinetic energies, which has the same value at low speeds, in the distribution function which is the same, except for that kinetic energy, we have the same distribution at low speeds. This of course should also be clear from plots of the MSD. To draw these we must first integrate the function in order to get a value for the normalization constant C.

2.3 Normalization

Because the numbers that come across in the MSD (especially in a relativistic variant, where the speed of light is all around) are quite large, I used a system of units in which the speed of light c, Boltzmann's constant k and the mass of the considered particle m are all chosen to equal 1. That is in order to make the numerical integration become possible by the program used and nothing is hereby lost in the physics.

The numerical integration is carried out using Mathematica. A dependance of C on the temperature is therefore not too clear, because we only get numerical values for different temperatures. Using the normalization constants, plots of different temperatures can be made (Fig 3). In this plot the temperatures are in proportion 1:2:10:20:200 with increasing temperatures having peaks more and more to the right. A few things can be clearly seen from the plot.

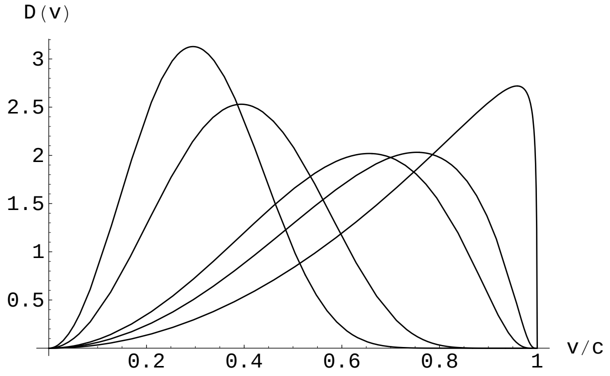

Figure 3: MSD for relativistic expressions of the kinetic energy. The units are such that $c = k = m = 1$, but the distributions are normalized.

3 Properties of the Relativistic MSD and Comparison

In contrast with the classical MSD the peaks get higher after a certain value for T. Can we understand this? If we take a close look at the plot we see that the left tail of the MSD is always pretty much the same as in the classical case. It is only in the regime of higher v where we notice a difference. For ever higher temperatures the particles must all have a very high speed, because of their thermal energy. The particles with a speed higher than a certain value v_1 must also have a speed smaller than c. Therefore the number of particles in the interval $v_1 < v < c$ must be higher than expected for the classic case. The speeds 'pile up'. The peak value gets ever closer to the speed of light, but will never reach it, no matter how high the temperature.

3.1 Limits of the Distribution

Now we will take a closer look at both ends of the distribution and see what changes if relativistic effects are taken into account.

For low temperatures the shape looks pretty much the same as in Fig 1 as was expected from the calculations. At low speeds the particles don't notice anything of their upper speed limit.

Let's take a look at the right tail of the distribution. In the classical case there was an exponential fall off, as could be seen from the equation too (Eq 7). From the relativistic variant you should expect the same behavior. To see this (and to verify that it still is the case for even bigger temperatures) we plot the very last part of Fig 3, for a temperature 20 times higher than the highest temperature in Fig 3 (not drawn in that plot, because of the big difference in scale). The result can be seen in Fig 4.

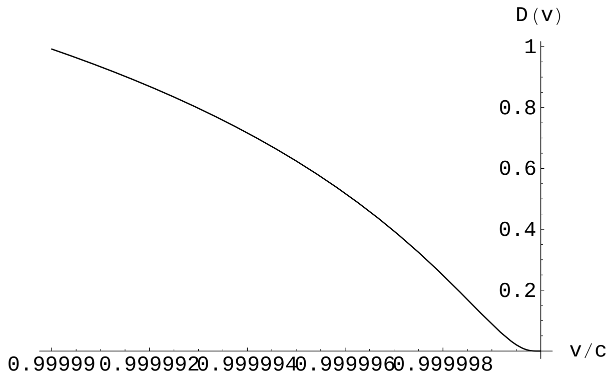

Figure 4: The right wing of the Maxwell Speed Distribution at extremely high temperatures.

The exponential fall off is very clear from this plot too. So Although it is very tightly packed, the exponential fall off will make sure that the number of particles with speeds too close to the speed of light will still be low.

3.2 Mean values of the speeds

Just as in section 1.1 there are three mean values of the speed. In the first place there is the root-mean-square speed. We can find this speed by putting the kinetic energy equal to the thermal energy, just like in Eq 1:

$$mc^2(\gamma(v_{rms}) - 1) = \frac{3}{2}kT \qquad (20)$$

Rearranging terms gives us:

$$v_{rms} = c\sqrt{1 - \left(\frac{3kT}{2mc^2} + 1\right)^{-2}} \qquad (21)$$

We can compare both dependencies of the root-mean-square speed on the temperature by plotting both the relativistic and the classical rms speed against temperature as is done in Fig 5. It can be eas-

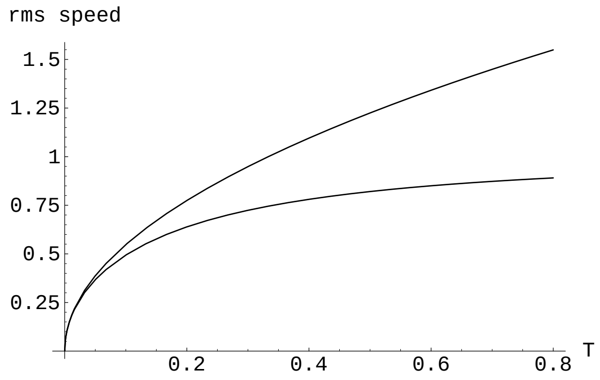

Figure 5: The classical (upper line) and relativistic (lower line) root-mean-square speeds as a function of temperature (in units where $c = k = m = 1$).

ily seen that for lower temperatures the rms speed is again the same and for higher speeds the relativistic correction gives values below c, while the usual calculation gives values possibly higher than the speed of light.

It will be obvious that similar statements hold for the speed at which the distribution function has its maximum (v_{max}) and for the average speed ($\overline{v}$) too. The former can be computed by the implicit equation

$$\frac{\partial}{\partial v} D(v_{max}) = 0 \tag{22}$$

The latter again is given by the integral

$$\overline{v} = 4\pi C \cdot \int_0^c v^3 \mathrm{Exp}\Big[\frac{m_0 c^2}{kT}\Big(1 - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Big)\Big] dv \tag{23}$$

Because of the required numerical integrations, it is not easily possible to become expressions for them as a function of temperature. Therefore only qualitative statements will be made.

In section 1.1 we saw that v_{max} was the smallest of three, followed up by $\overline{v}$ and the biggest value was for v_{rms}. De average speed can be seen as the 'center of mass' of the distribution. From Fig 3 it can be seen that v_{max} comes ever closer to the speed of light, and the left tail of the function is becoming bigger. That means that for very high temperatures $v_{max} > \overline{v}$. The order of the mean values thus changes too if relativistic effects are taken into account.

4 Overview

In this paper the Maxwell Speed Distribution is transformed in such a way that it obeys the special theory of relativity. That is, no speeds bigger than the speed of light are allowed for particles in a gas. To be able to do so the expression for kinetic energy as used in the original MSD is replaced by an appropriate relativistic expression.

With that expression a relativistic form of the MSD is derived. The distribution function could not be integrated analytically anymore, so numerical methods are used. After determination of the integration constant for several temperatures, plots are made of the MSD for those temperatures.

Comparison with the original distribution function learns us (both analytically and graphically) that for low temperatures (and hence low speeds for most particles) the relativistic form gives the same shape for the distribution function. For higher temperatures (even that high, that it may be questioned whether it happens anywhere in the universe at all) the MSD is really of a quite different shape if relativistic effects are taken into account. It looks like you push the 'old' distribution against a wall placed at $v = c$.

Is this at all physically relevant? For most of the purposes in everyday (astro-)physics it is not that important. It is more important for particles with smaller masses, because they have bigger speeds at the same kinetic energy. So if, in the end, the neutrino turns out to have a tiny little mass (which is, as far as I know, still possible), then we may find a place to use all this. Besides the physical relevance I found it interesting to see what happens at that enormously high temperatures where, in the classical sense, it was possible for particles to have a speed exceeding the speed of light.